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ARTICLE TEMPLATE

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ABSTRACT
In this work, an observer for a linear time-varying system with delayed measurements is developed. The delay is assumed to be unknown, bounded, and it can be time-varying with no restriction on its rate of change. The observer uses auxiliary signals related to the constructibility Gramian of the system and it contains nonlinearities that provide a uniform fixed-time convergence to a bounded region in the estimation error coordinates. This means that the convergence time can be bounded by a positive constant which is independent from the initial conditions and the initial time. This property is new for the addressed class of systems. The ultimate bound of the estimation error depends on the maximum difference between the nominal output and the delayed one, and not directly on the delay size or its time derivative. These properties are illustrated in a numerical simulation.

KEYWORDS
Observer; Delayed output; Linear Time-varying Systems

1. Introduction

The internal state estimation of a given system is one of the basic tasks in automatic control theory (Astolfi & Marconi, 2008; Besançon, 2007; Crassidis & Junkins, 2012; Meurer, Graichen, & Gilles, 2005). To develop such a task, an input-output information of the system is needed. If the information is carried through a network, or transmitted over long distances, it will be available but with a delay. The design of observers to perform the estimation using delayed data has been a topic of recent interest (Assche, Ahmed-Ali, Hann, & Lamnabhi-Lagarrigue, 2011; Cacace, Conte, Germani, & Palombo, 2017; Cacace, Germani, & Manes, 2010; Khosravian, Trumpf, Mahony, & Hamel, 2016; Vafaei & Yazdanpanah, 2016). In the case of linear time invariant system (LTI), the approaches based on delayed output error injection have been developed. This strategy can be used, among others, in the case of a known constant delay (Besançon, Georges, & Benayache, 2007), an unknown but constant delay (Cacace et al., 2017), or a time-varying known delay (Cacace et al., 2010), (Fridman, 2014a)[Sec. 5.2],(Kruszewski, Jiang, Fridman, Richard, & Toguyeni, 2012). These re-

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sults can also be applied for a certain class of nonlinear systems, as it is shown in the cited works, but there are also specialized works that study this type of systems. Examples of these works are (Anguelova & Wennberg, 2008) where conditions for the identifiability of constant delays in nonlinear systems are given, also (Ibrir, 2009) where an observer is proposed for nonlinear systems in triangular form and the observer gain is adapted by solving an algebraic Riccati equation depending on a dynamic parameter, or (Ghanes, Leon, & Barbot, 2013) where the authors present an observer that provides a bounded estimation error in the presence of a time-varying, unknown, and bounded delay. The results of this last reference are the closest in essence to the objectives of this work.

Most of these works require to check a linear matrix inequality (LMI) to establish the convergence of the observer. Application of LMIs is common in the study of time-delay systems (Fridman, 2014b; Sun & Chen, 2017). However, an analogue result for linear time-varying (LTV) systems seems to be not available. This can be related to the scarce results about stability of time-delay LTV systems and the difficulties that arise in their study. Among such works, one can find (Alaviani, 2009) where conditions are provided in terms of LMIs involving time-varying matrices, the stability conditions for a class of positive system are given in (Mazenc & Malisoff, 2016), or in (Zhou & Egorov, 2016) where the stability conditions are stated in terms of a Lyapunov function for the nominal case, i.e. without delay.

In this note, a LTV plant with delayed measurements is considered. The delay is assumed to be time-varying and bounded, and no restriction over its speed of variation will be imposed. The upper and lower bounds for the delay are also assumed to be unknown, and they are not needed for the design. Also, it is required that the system with the undelayed output be uniformly completely observable. Under these assumptions, an observer which provides fixed-time convergence of the estimation error to a ball is proposed. An important difference between the approaches mentioned previously and what we are proposing is that the effect of the delay is not introduced in the error dynamics. This allows to study the observer convergence by means of standard techniques used to analyze LTV systems. The observer also includes a nonlinearity, based on a Gramian-like constructions, which is responsible for the accelerated rate of convergence. Additionally, the ball to which the estimation error converges, depends on the difference between the delayed and the nominal output, and not directly on the size of the delay or its time derivative, making the approach suitable for delays that are large in the time scale of the system. Despite the fact that the design of finite and fixed-time convergent observers has been on focus recently (Andrieu, Praly, & Astolfi, 2008; Cruz-Zavala & Moreno, 2016; Cruz-Zavala, Moreno, & Fridman, 2012, 2011; Lopez-Ramirez, Efimov, Polyakov, & Perruquetti, 2016; Polyakov, 2012; Ríos & Teel, 2016), their advantages in the case of time-delay systems are not fully investigated yet. A preliminary version of this work can be found in (Rueda-Escobedo, Ushirobira, Efimov, & Moreno, 2018). The main difference with the previous version is the extension to LTV systems and a refinement on the ultimate bound for the estimation error.

The paper outline is as follows. In Section 2, the class of system under study and the problem statement are given. Some preliminary results regarding Riccati differential equations are discussed in Section 3. The observer structure and its properties are given in Section 4. The analysis of the estimation error and the proof of the results are developed in Section 5. The properties of the observer are illustrated in a numerical example in Section 6. Some auxiliary lemmas are established in the Appendix.

Notation: Let $\mathbb{R}_{>}0$ and $\mathbb{R}_{\geq}0$ be the sets of positive and non-negative real numbers, respectively; $\mathbb{R}^n$ denotes the real Euclidean space of dimension n; $\mathbb{R}^{n\times m}$ is the space of real matrices of n rows and m columns and $\mathbf{I}_n$ denotes the identity matrix of $\mathbb{R}^{n\times n}$. For $x \in \mathbb{R}^n$ and $p \geq 1$, $\|x\|_p$ is the p-norm, defined as $\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$. For $A \in \mathbb{R}^{n\times n}$, $\|A\|_p$ means the induced matrix norm. Whenever the subscript p is omitted, $\|\cdot\|$ refers to the Euclidean norm, i.e. $p = 2$. For two symmetric matrices Q_1, $Q_2 \in \mathbb{R}^{n\times n}$, $Q_1 > Q_2$ $\big(Q_1 \geq Q_2\big)$ means that $Q_1 - Q_2$ is positive (semi-) definite. Given $x \in \mathbb{R}$ and $p \in \mathbb{R}_{\geq 0}$, $\lceil x \rfloor^p$ denotes $|x|^p \mathrm{sign}(x)$, if the exponent is omitted, it correspond to $p = 1$. For $x \in \mathbb{R}^n$, $\lceil x \rfloor^p$ is understood element-wise.

2. Problem statement and motivation

In this note the state estimation of a linear time-varying system with delayed output will be investigated. To begin with, let us consider the following nominal system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ \bar{y}(t) &= C(t)x(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^m$, and $u \in \mathbb{R}^r$ are the state, the output, and the input vectors, respectively. The matrices $A(t)$, $B(t)$, and $C(t)$ are assumed to be known, piecewise continuous in t, and uniformly bounded in their norm. The state transition matrix associated to $A(t)$, which maps $x(t_1) \to x(t_2)$ in the absence of inputs, will be denoted by $\Phi(t_2, t_1)$. In the following $\tau(t) : \mathbb{R} \to [0, \bar{\tau}]$ will denote the delay and $\bar{\tau}$ its upper bound. The state, input, and system matrices are assumed to be defined over the interval $[t_0 - \bar{\tau}, \infty)$, where $t_0 \geq 0$ represent the process start time. Two kinds of delayed output will be recognized: when the delay affect both, the output matrix $C(t)$ and the state, and when the delay only appears in the state. If only the output matrix is delayed, one can rename it as $\bar{C}(t) = C(t - \tau(t))$, and address the problem as if there were no delay. We will focus our attention in the first case, when the output is completely delayed, since the second one can be treated by just slightly changing the notation. Then, the system of interest is as follows:

$$\begin{aligned} \dot{x}(t) &= A(t)\, x(t) + B(t)u(t), \\ y(t) &= C\big(t - \tau(t)\big)x\big(t - \tau(t)\big). \end{aligned} \tag{2}$$

In order to estimate the state of (2), we would ask the following:

Assumption 2.1. *There exist positive constants $T > 0$, $\alpha_1 \geq \alpha_2 > 0$ such that*

$$\alpha_1 \mathbf{I}_n \geq \mathcal{W}(t, t-T) := \int_{t-T}^{t} \Phi^\top(s,t) C^\top(s) C(s) \Phi(s,t)\, ds \geq \alpha_2 \mathbf{I}_n$$

for all $t \in [t_0 + T, \infty)$, that is, the pair $(A(t), C(t))$ is uniformly completely constructible.

Remark 1. Constructibility is related with the ability to reconstruct the current state from past measurements, whereas observability correspond to the reconstruction of the initial conditions from future data ("Chapter 4 Observability/Constructibility", 1977). In linear continuous-time systems, both properties are equivalent since $\Phi^\top(t + T, t)\mathcal{W}(t + T, t)\Phi(t + T, t)$ is the observability Gramian.

One way to approach the posed problem is to propose a delayed observer of the form

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) - L(t)\Big(C\big(t-\tau(t)\big)\hat{x}\big(t-\tau(t)\big) - y(t)\Big), \tag{3}$$

where $L(t)$ is a piecewise continuous function denoting the observer gain. Defining the estimation error as $e(t) = \hat{x}(t) - x(t)$, its dynamics results in

$$\dot{e}(t) = A(t)e(t) - L(t)C\big(t-\tau(t)\big)e\big(t-\tau(t)\big). \tag{4}$$

If one is able to design $L(t)$ in the nominal case, it can be expected that this approach work for sufficient small delay. Notice that (3) requires the knowledge of the delay in order to be implemented. The described method has been successfully applied for LTI system with constant delay (Besançon et al., 2007), and for time-varying one (Léchappé, Moulay, & Plestan, 2016). In both cases it is required that $\bar{\tau}$ satisfies some size restriction. If the delay is larger, the authors of (Besançon et al., 2007) has shown that the estimation can be handled by a chain of n observers. Each of the observers is oriented on treatment of an equivalent delay of τ/n, then if n is big enough, the scheme will provide an accurate estimate.

Another manner to approach the problem is to apply an undelayed output error injection, resulting in:

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) - L(t)\big(C(t)\hat{x}(t) - y(t)\big), \tag{5}$$

which does no require the value of $\tau(t)$, but imposes another source of inaccuracy. The error dynamics produced by this approach revels that it will depend on the difference between $y(t)$ and $\bar{y}(t)$:

$$\begin{aligned}\dot{e}(t) &= \big(A(t) - L(t)C(t)\big)e(t) + L(t)\big(y(t) - C(t)x(t)\big)\\ &= \big(A(t) - L(t)C(t)\big)e(t) + L(t)\big(y(t) - \bar{y}(t)\big)\end{aligned} \tag{6}$$

Then, the boundedness of $\|y(t)-\bar{y}(t)\|$ implies the boundedness of $\|e(t)\|$. This happens if, for example, the matrix $A(t)$ defines a uniformly asymptotically stable motion. Since the delay is not needed in this case, it can be assumed uncertain, time-varying and bounded. The ultimate bound of the error is given in the following lemma:

Lemma 2.2. *Let $P(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ be a continuously differentiable matrix function, $Q(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ be a piecewise continuous matrix function, $p_{L,1}\boldsymbol{I}_n \geq P_L(t) \geq p_{L,2}\boldsymbol{I}_n$ and $q_{L,1}\boldsymbol{I}_n \geq Q_L(t) \geq q_{L,2}\boldsymbol{I}_n$ with positive constants $p_{L,1} \geq p_{L,2} > 0$ and $q_{L,1} \geq q_{L,2} > 0$, and they satisfy the differential Lyapunov inequality for any $t_0 \in \mathbb{R}_{\geq 0}$ and all $t \geq t_0$*

$$\dot{P}_L(t) + P_L(t)\big(A(t) - L(t)C(t)\big) + \big(A(t) - L(t)C(t)\big)^\top P_L(t) \leq -Q_L(t)$$

for a given piecewise continuous and bounded matrix function $L(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times r}$. Then in (6) the error $e(t)$ stays bounded for all $t \geq t_0$ and

$$\lim_{t\to\infty} \|e(t)\| \leq 2\sqrt{\frac{p_{L,1}}{p_{L,2}}} \cdot \frac{p_{L,1}}{q_{L,2}} \sup_{t\geq t_0} \|L(t)\|\,\|y(t) - \bar{y}(t)\|. \tag{7}$$

Proof. Consider the Lyapunov function candidate $V(t,e) = e^\top P(t)e$. Its derivative along (6) can be bounded as follows:

$$
\begin{aligned}
\dot{V}(t) &\le -e^\top(t)Q(t)e(t) + 2\,e^\top(t)P(t)L(t)\big(y(t)-\bar{y}(t)\big)\\
&\le -\frac{1}{2}q_{L,2}\|e(t)\|^2 + 2\frac{p_{L,1}^2}{q_{L,2}}\big\|L(t)\big\|^2\big\|y(t)-\bar{y}(t)\big\|^2\\
&\le -\frac{q_{L,2}}{2\,p_{L,1}}V(t) + 2\frac{p_{L,1}^2}{q_{L,2}}\big\|L(t)\big\|^2\big\|y(t)-\bar{y}(t)\big\|^2,
\end{aligned}
$$

then

$$
V(t) \le V(t_0)\exp\Big(-\frac{q_{L,2}}{2\,p_{L,1}}(t-t_0)\Big) + \frac{4\,p_{L,1}^3}{q_{L,2}^2}\sup_{t\ge t_0}\|L(t)\|^2\|y(t)-\bar{y}(t)\|^2
$$

$$
\|e(t)\| \le \sqrt{\frac{p_{L,1}}{p_{L,2}}}\|e(t_0)\|\exp\Big(-\frac{q_{L,2}}{4\,p_{L,1}}(t-t_0)\Big) + 2\sqrt{\frac{p_{L,1}}{p_{L,2}}}\cdot\frac{p_{L,1}}{q_{L,2}}\sup_{t\ge t_0}\|L(t)\|\,\|y(t)-\bar{y}(t)\|.
$$

From the last expression, the bound follows. □

Both of the described methods has some disadvantages. In the case (3), not only $\tau(t)$ has to be known, but the design of $L(t)$ can be really challenging, being this particularly true for LTV systems. In the second case (5), the delay can be unknown at the price of having a bounded error, and if one require to approach this bound faster, such bound will increase because this can only be achieved by increasing $L(t)$. Based on these observations the second approach seems to be more convenient, first, because there is no an extension of (3) to the LTV case, and second, because it does not require precise knowledge of the delay. To alleviate the problem w.r.t. the convergence rate, in this note, we will provide a methodology to modify (5) by introducing some nonlinearities in order to obtain a uniform rate of convergence, that is, the capability of reaching a bounded region of $e(t) = 0$ uniformly in t_0 and in the initial error. This will be done under the following hypothesis:

Assumption 2.3. *The input is known and uniformly bounded, i.e.,* $\|u(t)\| \le u_M < \infty$ *for all* $t \ge t_0$*.*

Assumption 2.4. *The output of the system is a Lipschitz function of time, that is, there exist a constant* $\gamma > 0$ *such that*

$$
\big\|y(t_1) - y(t_2)\big\| \le \gamma\,\big|t_2 - t_1\big| \quad \forall\, t_1, t_2 \ge t_0 - \bar{\tau}.
$$

Remark 2. For example, the property in Assumption 2.4 is obtained if the matrix $A(t)$ describes an uniformly asymptotically stable motion. In such case, and because the state remains bounded, the difference $\|y(t)-\bar{y}(t)\|$ can be bounded independently of $\bar{\tau}$. Also, in the case of a LTI system with one pole at zero and the rest in the open left-half complex plane, where the system is marginally stable, Assumption 2.4 is satisfied.

Remark 3. If a system satisfy the Assumption 2.4, then $\|y(t)-\bar{y}(t)\| \le \gamma\,\tau(t) \le \gamma\,\bar{\tau}$.

Assumption 2.3 and 2.4 are required in order to keep the error $\|y(t)-\bar{y}(t)\|$ bounded. On the other hand, if $A(t)$ describes an uniformly asymptotically stable motion, one can use a copy of the plant as an observer, without using any kind of correction term. In such approach, there is no control over the convergence velocity, and one has to relay on the intrinsic properties inscribed in $A(t)$, whereas an objective of this work is to increase the rate of convergence. Nevertheless, the price to pay is a bounded error. Finally, we want to remark that, in the case of delay-independent stability, uniform asymptotic stability is necessary for both, LTI (Fridman, 2014b) and LTV (Zhou & Egorov, 2016) systems. In the case of delay-dependent stability, and for LTI systems, asymptotic stability might reduce the difficulty in finding L (Besançon et al., 2007).

3. Preliminaries: Riccati differential equations

In this section some properties about a Riccati differential equation (RDE), related to observation, will be discussed together with some properties related to the uniform asymptotic stability of LTV systems. The solution of the aforementioned RDE, which can be computed on-line, will be used to propose a correction term for the observer.

Consider the following RDE:

$$\begin{aligned} \dot{N}(t) &= -A^\top(t)N(t) - N(t)A(t) - N(t)\Theta(t)N(t) + C^\top(t)C(t), \\ N(t_0) &= N_0 = N_0^\top \geq 0, \end{aligned} \tag{8}$$

with a piecewise continuous matrix function $\Theta(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ satisfying $\theta_1 \mathbf{I}_n \geq \Theta(t) \geq \theta_2 \mathbf{I}_n$ for some positive constants $\theta_1 \geq \theta_2 > 0$. This RDE is commonly associated to the Kalman-Bucy filter. It has been proved that under Assumption 2.1 and the bounds imposed to $\Theta(t)$, $N(t)$ is uniformly bounded and invertible (Anderson, 1971; Bucy, 1972). In particular, it is shown in (Anderson, 1971) that $N(t)$ and $\Phi^\top(t_0,t)N_0\Phi(t_0,t) + \mathcal{W}(t,t_0)$ share the same null space. Furthermore, in (Bucy, 1972) the following bounds are provided:

$$\begin{aligned} N(t) &\leq \mathcal{C}^{-1}(t, t-T) + \lambda\, \mathcal{W}(t, t-T), \\ N(t) &\geq \Big(\mathcal{W}^{-1}(t, t-T) + \lambda\, \mathcal{C}(t, t-T)\Big)^{-1}, \end{aligned} \tag{9}$$

for all $t \geq t_0 + T$, where $\mathcal{C}$ is defined as

$$\mathcal{C}(t_2, t_1) := \int_{t_1}^{t_2} \Phi(t_2, s)\Theta(s)\Phi^\top(t_2, s)\mathrm{d}s.$$

Given θ_1 and θ_2, there always exist $\beta_1 \geq \beta_2 > 0$ such that $\beta_1 \mathbf{I}_n \geq \mathcal{C}(t, t-T) \geq \beta_2 \mathbf{I}_n$. Taking $\alpha_1 \geq \alpha_2 > 0$ as in Assumption 2.1, λ can be chosen as $\lambda = n^2(\alpha_1\beta_1)/(\alpha_2\beta_2)$. In particular, if N_0 is taken definite positive, $H(t) := N^{-1}(t)$ exists for all $t \geq t_0$ and satisfies:

$$\dot{H}(t) = H(t)A^\top(t) + A(t)H(t) - H(t)C^\top(t)C(t)H(t) + \Theta(t),\ H(t_0) = N_0^{-1}.$$

This dynamics follows from deriving the relation $H(t)N(t) = \mathbf{I}_n$, which results in $\dot{H}(t) = -H(t)\dot{N}(t)H(t)$. These two matrices, $N(t)$ and $H(t)$, will be of interest along the note.

Remark 4. Although bounds (9) are of particular importance to establish stability results, the evaluations that they provide are, in general, very conservative.

Now, consider the auxiliary function $\bar{\psi}(t) = N(t)x(t)$, which being calculated due to the invertibility of $N(t)$ provides an estimate for $x(t)$ immediately. To compute $\bar{\psi}(t)$, let us write its time derivative. When there is no delay, this results in

$$\begin{aligned}\dot{\bar{\psi}}(t) &= \dot{N}(t)x(t) + N(t)\dot{x}(t)\\ &= -\Big(A^\top(t) + N(t)\Theta(t)\Big)\bar{\psi}(t) + N(t)B(t)u(t) + C^\top(t)\bar{y}(t).\end{aligned} \tag{10}$$

To preserve the proposed relationship, $\bar{\psi}(t_0)$ should be taken as $N_0x(t_0)$, which require the initial condition of the system. Since $\bar{y}(t)$ and $x(t_0)$ are not available, we propose to compute an estimate $\psi(t)$ of the auxiliary function $\bar{\psi}(t)$ as

$$\dot{\psi}(t) = -\Big(A^\top(t) + N(t)\Theta(t)\Big)\psi(t) + N(t)B(t)u(t) + C^\top(t)y(t),\ \psi(t_0) = \psi_0, \tag{11}$$

where the available output is used, and the initial condition is left free. Now, consider the error $\Delta_\psi(t) = \bar{\psi}(t) - \psi(t)$ whose dynamics yields

$$\dot{\Delta}_\psi(t) = -\Big(A^\top(t) + N(t)\Theta(t)\Big)\Delta_\psi(t) + C^\top(t)\Big(\bar{y}(t) - y(t)\Big). \tag{12}$$

Then, if the matrix $-\Big(A^\top(t) + N(t)\Theta(t)\Big)$ is uniformly asymptotically stable, $\Delta_\psi(t)$ will remain bounded, meaning that $\psi(t)$ can be used instead of $\bar{\psi}(t)$ despite the lack of correct information.

Lemma 3.1. *Let* $\eta_1\mathbf{I}_n \geq N(t) \geq \eta_2\mathbf{I}_n$. *Then*

$$\lim_{t\to\infty} \|\Delta_\psi(t)\| \leq \sqrt{\frac{\eta_1}{\eta_2\theta_2}} \sup_{t\geq t_0} \|\bar{y}(t) - y(t)\|. \tag{13}$$

Proof. Consider as a Lyapunov function candidate $V(\Delta_\psi, t) = \Delta_\psi^\top H(t)\Delta_\psi$, which satisfies $\frac{1}{\eta_2}\|\Delta_\psi\|^2 \geq V(\Delta_\psi, t) \geq \frac{1}{\eta_1}\|\Delta_\psi\|^2$. Its derivative along (12) results in

$$\begin{aligned}\dot{V}(t) &= -\Delta_\psi^\top(t)\Big(H(t)C^\top(t)C(t)H(t) + \Theta(t)\Big)\Delta_\psi(t) + 2\,\Delta_\psi^\top(t)H(t)C^\top(t)\Big(\bar{y}(t) - y(t)\Big)\\ &\leq -\theta_2\|\Delta_\psi(t)\|^2 - \|C(t)H(t)\Delta_\psi(t)\|^2 + 2\,\|C(t)H(t)\Delta_{\psi(t)}\|\,\|\bar{y}(t) - y(t)\|\\ &\leq -\theta_2\|\Delta_\psi(t)\|^2 + \|\bar{y}(t) - y(t)\|^2\\ &\leq -\eta_2\theta_2V(t) + \|\bar{y}(t) - y(t)\|^2.\end{aligned}$$

The last inequality implies that

$$V(t) \leq V(t_0)\exp\big(-\eta_2\theta_2(t - t_0)\big) + \frac{1}{\eta_2\theta_2}\sup_{t\geq t_0}\|\bar{y}(t) - y(t)\|^2$$

$$\|\Delta_\psi(t)\| \leq \sqrt{\frac{\eta_1}{\eta_2}}\|\Delta_\psi(t_0)\|\exp\Big(-\frac{1}{2}\eta_2\theta_2(t - t_0)\Big) + \sqrt{\frac{\eta_1}{\eta_2\theta_2}}\sup_{t\geq t_0}\|\bar{y}(t) - y(t)\|.$$

Taking the limit when $t \to \infty$, the bound of the lemma follows. $\square$

In the next section, the term $N(t)\hat{x}(t) - \psi(t)$ will be used to enhance the convergence rate.

4. Main Result

The undelayed observer (5) will be taken as a starting point for the nonlinear observer proposed in this section. The results developed in Section 3 will be used to strengthen its convergence rate. Denote by $\hat{x}(t)$ the estimate of $x(t)$, and define the estimation error as $e(t) = \hat{x}(t) - x(t)$. Recalling that $\psi(t) = N(t)x(t) - \Delta_\psi(t)$, we remark that $N(t)\hat{x}(t) - \psi(t) = N(t)e(t) + \Delta_\psi(t)$. Then, the term $N(t)\hat{x}(t) - \psi(t)$ carries an information about the estimation error and the matrix $N(t)$ is positive definite. Following this observation, we propose the following system as an observer for (2):

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) - H(t)C^\top(t)\Big(C(t)\hat{x}(t) - y(t)\Big) \\ &\quad - \Lambda \left\lceil N(t)\hat{x}(t) - \psi(t) \right\rfloor^p, \end{aligned} \tag{14}$$

where the auxiliary signals $N(t)$, $\psi(t)$, and $H(t)$ are computed following

$$\dot{N}(t) = -A^\top(t)N(t) - N(t)A(t) - N(t)\Theta(t)N(t) + C^\top(t)C(t), \tag{15}$$

$$\dot{\psi}(t) = -\Big(A^\top(t) + N(t)\Theta(t)\Big)\psi(t) + N(t)B(t)u(t) + C^\top(t)y(t), \tag{16}$$

$$\dot{H}(t) = H(t)A^\top(t) + A(t)H(t) - H(t)C^\top(t)C(t)H(t) + \Theta(t), \tag{17}$$

and $p > 1$ is an exponent to be chosen, whereas the matrix $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ with $\lambda_i > 0$ contains tuning parameters. In (15)-(17), $\Theta(t)$ has to satisfy the bounds $\theta_1 \mathbf{I}_n \geq \Theta(t) \geq \theta_2 \mathbf{I}_n$ for some positive θ_1 and θ_2. Finally, the initial condition of N, N_0, has to be selected positive definite, and $H(t_0) = N_0^{-1}$. The initial conditions for $\hat{x}$ and ψ are free.

Denote by $\tilde{y}(t) = \bar{y}(t) - y(t)$. The error dynamics produced by implementing (14) results in

$$\dot{e}(t) = \Big(A(t) - H(t)C^\top(t)C(t)\Big)e(t) + H(t)C^\top(t)\tilde{y}(t) - \Lambda\left\lceil N(t)e(t) + \Delta_\psi(t)\right\rfloor^p. \tag{18}$$

Theorem 4.1. *Let Assumptions 2.1, 2.3, and 2.4 be satisfied, $p > 1$, $\theta_1 \mathbf{I}_n \geq \Theta(t) \geq \theta_2 \mathbf{I}_n$, $\Lambda > 0$, and $N_0 > 0$. Then there exist η_1 and η_2 satisfying $\eta_1 \mathbf{I}_n \geq N(t) \geq \eta_2 \mathbf{I}_n$. Furthermore, the estimation error converge uniformly in fixed-time to the region given by*

$$\|e(t)\| \leq \sqrt{\frac{\rho}{\eta_2}} \qquad \forall\, t \geq t_0 + T_\star, \tag{19}$$

where ρ is the unique positive root of

$$\mathrm{P}(\mathrm{v}) = \theta_2 \frac{\eta_2^2}{\eta_1} \mathrm{v} + 2\,\lambda_m n^{\frac{1-p}{2}} \kappa_1 \frac{\eta_2^{p+1}}{\eta_1^{\frac{p+1}{2}}} \mathrm{v}^{\frac{p+1}{2}} - \frac{\gamma^2 \bar{\tau}^2}{q} - \frac{2\,\lambda_m \kappa_2}{q} \sup_{t \geq t_0} \|\Delta_\psi(t)\|^{p+1}, \tag{20}$$

with $q \in (0,1)$, $\lambda_m = \min_{1\le i\le n} \lambda_i$, κ_1 and κ_2 as in Corollary A.2, and where the convergence time is at most $T_\star$ and is given by:

$$T_\star = \frac{2^{\frac{\eta_1}{\eta_2}}}{\theta_2\eta_2(p-1)(1-q)} \ln\left[1 + \frac{\theta_2}{2\,\lambda_m n^{\frac{1-p}{2}}\kappa_1\eta_2^{\frac{p-1}{2}}}\left(\frac{\eta_1}{\eta_2}\right)^{\frac{p-1}{2}}\right].$$

The uniform fixed-time convergence follows from the fact that $T_\star$ bounds the convergence time for any initial error and initial time and $T_\star$ does not depend on them. The structure of $T_\star$ reveals that the convergence time can be reduced by increasing p, making the condition number of $N(t)$, (η_1/η_2), smaller and by increasing λ_m. Although θ_2 seems to be a free parameters, changing it affects both, η_1 and η_2. The proof of Theorem 4.1 is given in Section 5.

Remark 5. If $N_0 \ge 0$ or $H(t_0) \ne N_0^{-1}$ for some reason, there appears an error that vanishes exponentially fast (Kalman, 1960)[Sec. 7], leaving the qualitative result of Theorem 4.1 unaltered.

Corollary 4.2. *Make the same assumptions as in Theorem 4.1, but with $p = 3$. Then*

$$\|e(t)\| \le \frac{1}{\eta_2}\left(\frac{n}{q}\right)^{1/4}\sqrt{\frac{\eta_1}{\eta_2}}\left(1.786\frac{\sqrt{\gamma\,\bar\tau}}{\lambda_m^{1/4}} + 1.211\sup_{t\ge t_0}\|\Delta_\psi(t)\|\right) \quad \forall\, t \ge t_0 + T_\star, \qquad (21)$$

where $q \in (0,1)$, $\lambda_m = \min_{1\le i\le n}\lambda_i$, and

$$T_\star = \frac{1}{\theta_2\eta_2(1-q)}\cdot\frac{\eta_1}{\eta_2}\ln\left[1 + 10.2\frac{n\,\theta_2}{\lambda_m\eta_2}\cdot\frac{\eta_1}{\eta_2}\right].$$

Proof. With $p = 3$ and using the values of κ_1 and κ_2 in Table A2, the polynomial (20) becomes

$$\mathrm{P}(\mathrm{v}) = \theta_2\frac{\eta_2^2}{\eta_1}\mathrm{v} + 0.0982\frac{\lambda_m\eta_2^4}{n\,\eta_1^2}\mathrm{v}^2 - \frac{\gamma^2\bar\tau^2}{q} - 0.2114\frac{\lambda_m}{q}\sup_{t\ge t_0}\|\Delta_\psi(t)\|^4.$$

A root of this polynomial is

$$\rho = \frac{5.092}{\lambda_m\eta_2}\left(\frac{\eta_1}{\eta_2}\right)\left(-n\,\theta_2 + \sqrt{\frac{n}{q}}\sqrt{n\,q\,\theta_2^2 + 0.3928\,\lambda_m\gamma^2\bar\tau^2 + 0.08304\,\lambda_m^2\sup_{t\ge t_0}\|\Delta_\psi(t)\|^4}\right).$$

Since the 1-norm is greater that the Euclidean norm (2-norm), we have

$$\rho \le \sqrt{\frac{n}{q}}\cdot\frac{\eta_1}{\eta_2^2}\left(3.19\,\frac{\gamma\,\bar\tau}{\lambda_m^{1/2}} + 1.467\sup_{t\ge t_0}\|\Delta_\psi(t)\|^2\right).$$

Then, the norm of the error satisfies:

$$\begin{aligned}\|e(t)\| &\le \tfrac{1}{\eta_2}\left(\tfrac{n}{q}\right)^{1/4}\sqrt{\tfrac{\eta_1}{\eta_2}\left(3.19\tfrac{\gamma\,\bar{\tau}}{\lambda_m^{1/2}}+1.467\sup_{t\ge t_0}\|\Delta_\psi(t)\|^2\right)} \\ &\le \tfrac{1}{\eta_2}\left(\tfrac{n}{q}\right)^{1/4}\sqrt{\tfrac{\eta_1}{\eta_2}\left(1.786\tfrac{\sqrt{\gamma\,\bar{\tau}}}{\lambda_m^{1/4}}+1.211\sup_{t\ge t_0}\|\Delta_\psi(t)\|\right)}\end{aligned} \qquad \forall\, t\ge t_0+T_\star.$$

Replacing the values of p and κ_1 in the definition of $T_\star$ we obtain the given estimate. □

In Corollary 4.2 we find the root of (20) for a particular choice of p. This allows us to see how the size of the attraction region behaves in this case, and it can give us an intuition on how it probably behaves for other values of p. As can be seen in (21), the root can be decreased if η_2 increases, or if the condition number of $N(t)$ decreases. However, these terms cannot be directly adjusted. On the other hand, there are two terms in (20), one depending on $\bar{\tau}$ and other depending on $\Delta_\psi(t)$, that also affects the size of the attraction region. The term depending on $\bar{\tau}$ can be decreased by increasing λ_m. The other one can be decreased by increasing θ_2 as shown in (13). Then, the size of the final error can be modified by varying these two parameters, λ_m and θ_2.

5. Convergence analysis and proof of claims

Consider as a Lyapunov function candidate $V(e,t)=e^\top N(t)e$, which is a valid candidate given the existence of η_1 and η_2. Its derivative along (18) yields

$$\begin{aligned}\dot V(t) =& \, e^\top(t)\Big(-N(t)\Theta(t)N(t)-C^\top(t)C(t)\Big)e(t)+2\,e^\top(t)C^\top(t)\tilde y(t) \\ &-2\,e^\top(t)N(t)\Lambda\lceil N(t)e(t)+\Delta_\psi(t)\rfloor^p \\ \le& -e^\top(t)N(t)\Theta(t)N(t)e(t)-\|C(t)e(t)\|^2+2\,\|C(t)e(t)\|\,\|\tilde y(t)\| \\ &-2\,\lambda_m\,e^\top(t)N(t)\lceil N(t)e(t)+\Delta_\psi(t)\rfloor^p \\ \le& -e^\top(t)N(t)\Theta(t)N(t)e(t)-2\,\lambda_m\,e^\top(t)N(t)\lceil N(t)e(t)+\Delta_\psi(t)\rfloor^p+\|\tilde y(t)\|^2 \\ \le& -e^\top(t)N(t)\Theta(t)N(t)e(t)-2\,\lambda_m\,e^\top(t)N(t)\lceil N(t)e(t)+\Delta_\psi(t)\rfloor^p+\gamma^2\bar{\tau}^2.\end{aligned}$$

Using the bounds for $N(t)$, $\Theta(t)$ and the result of Corollary A.2, the time derivative of V can be estimated as

$$\begin{aligned}\dot V(t) &\le -\theta_2\eta_2^2\|e(t)\|^2-2\,\lambda_m\,n^{\frac{1-p}{2}}\kappa_1\big\|N(t)e(t)\big\|^{p+1}+2\,\lambda_m\kappa_2\|\Delta_\psi(t)\|^{p+1}+\gamma^2\bar{\tau}^2 \\ &\le -\theta_2\eta_2^2\|e(t)\|^2-2\,\lambda_m\,n^{\frac{1-p}{2}}\kappa_1\eta_2^{p+1}\|e(t)\|^{p+1}+2\,\lambda_m\kappa_2\|\Delta_\psi(t)\|^{p+1}+\gamma^2\bar{\tau}^2.\end{aligned}$$

κ_1 and κ_2 can be chosen following tables A1 and A2. Since $V(t)\le\eta_1\|e(t)\|^2$, the previous inequality can be transformed into a differential inequality in terms of $V(t)$:

$$\begin{aligned}\dot V(t) \le& -\theta_2\eta_2\left(\frac{\eta_2}{\eta_1}\right)V(t)-2\,\lambda_m n^{\frac{1-p}{2}}\kappa_1\eta_2^{\frac{p+1}{2}}\left(\frac{\eta_2}{\eta_1}\right)^{\frac{p+1}{2}}V^{\frac{p+1}{2}}(t) \\ &+2\,\lambda_m\kappa_2\|\Delta_\psi(t)\|^{p+1}+\gamma^2\bar{\tau}^2 \\ \le& -k_1\,V(t)-k_2\,V^{\frac{p+1}{2}}(t)+\Delta,\end{aligned}$$

with

$$\Delta := \gamma^2\bar{\tau}^2 + 2\,\lambda_m \kappa_2 \sup_{t\geq t_0} \|\Delta_\psi(t)\|^{p+1},$$
$$k_1 := \theta_2\eta_2 \left(\frac{\eta_2}{\eta_1}\right),$$
$$k_2 := 2\,\lambda_m n^{\frac{1-p}{2}} \kappa_1 \eta_2^{\frac{p+1}{2}} \left(\frac{\eta_2}{\eta_1}\right)^{\frac{p+1}{2}}.$$

The constant Δ is finite due to Lemma 3.1 and assumptions 2.3 and 2.4. Let $q \in (0,1)$ and denote by ρ the unique positive root[1] of $k_1\,v + k_2\,v^{\frac{p+1}{2}} - \frac{1}{q}\Delta$. Then, we have that

$$\dot{V}(t) \leq -(1-q)\left(k_1 V(t) + k_2 V^{\frac{p+1}{2}}(t)\right) < 0,\ V(t) \geq \rho.$$

Consider the differential equation $\dot{z}(t) = -(1-q)\big(k_1 z(t) + k_2 z^{\frac{p+1}{2}}(t)\big),\ z(t_0) \geq 0$. As in (Moreno, 2012, pp. 134), the change of variable $w(t) = \exp\big((1-q)k_1\,(t-t_0)\big)\,z(t)$ transform the equation into:

$$\dot{w}(t) = -(1-q)k_2 \exp\Big(-\frac{1}{2}(p-1)(1-q)k_1(t-t_0)\Big) w^{\frac{p+1}{2}}(t).$$

By separation of variable, the solution for $w(t)$ and $z(t)$ results in:

$$w(t) = \left[\frac{1}{w^{\frac{p-1}{2}}(t_0)} + \frac{k_2}{k_1}\left(1 - \exp\Big(-\frac{1}{2}(p-1)(1-q)k_1(t-t_0)\Big)\right)\right]^{-\frac{2}{p-1}},$$
$$z(t) = \left[\left(\frac{1}{z^{\frac{p-1}{2}}(t_0)} + \frac{k_2}{k_1}\right)\exp\Big(\frac{1}{2}(p-1)(1-q)k_1(t-t_0)\Big) - \frac{k_2}{k_1}\right]^{-\frac{2}{p-1}}.$$

Then, by the Comparison Lemma (Khalil, 2002, Lem. 3.4) we have

$$V(t) \leq \left[\left(\frac{1}{V^{\frac{p-1}{2}}(t_0)} + \frac{k_2}{k_1}\right)\mathbf{e}^{\frac{1}{2}(p-1)(1-q)k_1(t-t_0)} - \frac{k_2}{k_1}\right]^{-\frac{2}{p-1}}.$$

Without loss of generality, assume that $V(t_0) > \rho$. Then, $V(t) \leq \rho$ for

$$t - t_0 \geq \frac{2}{k_1(p-1)(1-q)} \ln\left[\left(\frac{V(t_0)}{\rho}\right)^{\frac{p-1}{2}} \frac{k_1 + k_2\rho^{\frac{p-1}{2}}}{k_1 + k_2 V^{\frac{p-1}{2}}(t_0)}\right]$$
$$\geq \frac{2}{k_1(p-1)(1-q)} \ln\left[\frac{k_1 + k_2\rho^{\frac{p-1}{2}}}{k_2\rho^{\frac{p-1}{2}} + k_1\left(\frac{\rho}{V(t_0)}\right)^{\frac{p-1}{2}}}\right].$$

[1]For $v \geq 0$, $k_1\,v + k_2\,v^{\frac{p+1}{2}}$ is a strict positive monotonic function of v, then its image is $\mathbb{R}_{\geq 0}$.

Taking the limit when $V(t_0) \to \infty$ it yields

$$
\begin{aligned}
t - t_0 &\geq \frac{2}{k_1(p-1)(1-q)} \ln \left[1 + \frac{k_1}{k_2 \rho^{\frac{p-1}{2}}} \right] \\
&= \frac{2^{\frac{\eta_1}{\eta_2}}}{\theta_2 \eta_2 (p-1)(1-q)} \mathrm{Ln} \left[1 + \frac{\theta_2}{2\,\lambda_m n^{\frac{1-p}{2}} \kappa_1 \eta_2^{\frac{p-1}{2}}} \left(\frac{\eta_1}{\eta_2} \right)^{\frac{p-1}{2}} \right] =: T_\star .
\end{aligned}
$$

This bound represents the amount of time which guarantees that the level set $V(t) \leq \rho$ is reached. This bound is finite and does not depend on the initial time nor on the initial value of V, then the level set is reached in finite time, uniformly in t_0 and in the initial condition. Finally, since $V(t) \geq \eta_2 \|e(t)\|^2$, $\|e(t)\| \leq \sqrt{\rho/\eta_2}$ for all $t \geq t_0 + T_\star$.

6. Numerical example

To exemplify the proposed observer, the following system is considered:

$$
\begin{aligned}
\dot{x}(t) &= \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A_{22}(t)$}} \\ 0 & & \end{array} \right] x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) = A(t)x(t) + b\,u(t), \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t-\tau(t)) = C\,x(t-\tau(t)) = x_1(t-\tau(t)),
\end{aligned}
$$

with $A_{22}(t) = -\omega(t)\omega^\top(t)$, $\omega^\top(t) = [\cos(3\,t),\, 1]$, $u(t) = \cos(5\,t) + 1$.

It is well known that if $\omega(t)$ is of persistent excitation, $-\omega(t)\omega^\top(t)$ describes a uniformly asymptotically stable motion (Anderson, 1977). Then, the dynamics of $x_2(t)$ and $x_3(t)$ is uniformly asymptotically stable. However, $x_1(t)$ integrates $x_2(t)$, resulting unbounded, but a Lipschitz time function, fulfilling Assumption 2.4. For the simulation, the initial conditions of the system are set in $x_1(t_0) = 0$, $x_2(t_0) = 10$, and $x_3(t_0) = 5$. For this example, the delay is selected as $\tau(t) = \alpha + 2\,\alpha\cos(6\,t)/3$ with alpha taking the values $\{0.1,\, 0.3,\, 0.6,\, 1.2\}$. In the design and implementation of the observer, the value of $\tau(t)$ is not needed. The observer parameters were chosen as follows:

$$
\Theta(t) = \mathrm{diag}\big(10,\, 6,\, 6\big),\ \Lambda = 30\,\mathbf{I}_3,\ p = 2,\ N_0 = \mathbf{I}_3.
$$

The initial conditions for the observer were set to zero. For these conditions, the norm of the estimation error for the different values of α is shown in Figure 1. It can be seen that the error bound grows with the size of α. This can be attributed to the fact that the difference $|y(t) - \bar{y}(t)|$ also grows with α, as is shown in Figure 2.

Now, to show the fixed-time convergence, the initial condition of the observer was increased to induce initial errors of 10^3, 10^5 and 10^7, while keeping $\alpha = 1.2$. The evolution of the estimation norm, under these circumstances, is shown in Figures 4 and 5 in a logarithmic scale. In Figure 4 the fast attraction it is shown. In this figure, a logarithmic scale is also used for the time axis since the convergence was really fast. In Figure 5 it is shown that the three trajectories reach the same region in almost the same time despite the difference between the orders of magnitude in the initial condition.

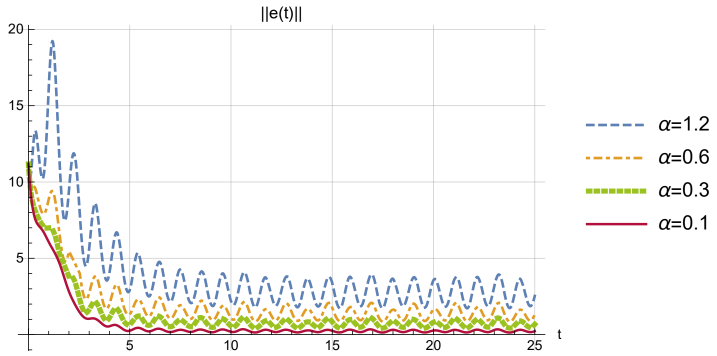

Figure 1. Evolution of the estimation error norm for the different values of α.

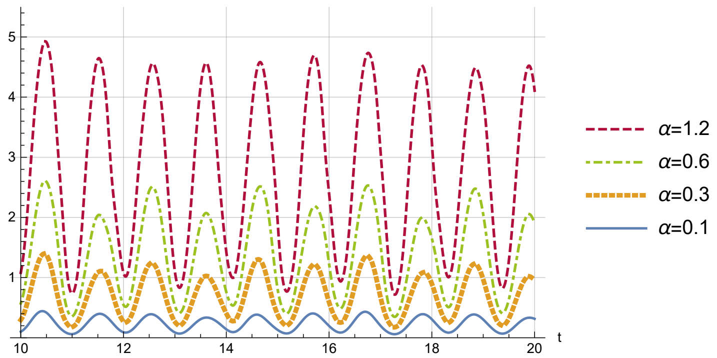

Figure 2. Plot of the difference $|y(t) - \bar{y}(t)|$. Increasing the value of α increases the value of $\bar{\tau}$.

In the previous scenario, the size of $|y(t) - \bar{y}(t)|$ increases with α given the marginal stability of the system. To shown how the observer behaves when the systems is uniformly asymptotically stable, we will consider only the dynamic of $x_2(t)$ and $x_3(t)$. The output will be $y(t) = x_2(t - \tau(t))$, leaving $\tau(t)$ as before. For this setting, the observer was configured as follows:

$$N_0 = \mathbf{I}_2, \quad \Theta(t) = 10\,\mathbf{I}_2, \quad \Lambda = 30\,\mathbf{I}_2, \quad p = 2.$$

The initial conditions for the observer state and $\psi(t)$ were set in zero. The difference between $y(t)$ and $\bar{y}(t)$ in this case is shown in Figure 7. Since the nominal output is bounded, the difference between it and the delayed one does not grow with α. This is reflected in the fact that the size of the convergence region is maintained despite the increase in α, as it is demonstrated in Figure 6. This contrasts with the previous situation where the error grew with the delay.

As before, to show the fixed-time convergence, the initial error was increased in orders of magnitude of 10^3, 10^5, and 10^7 while keeping $\alpha = 1.2$. In Figure 8 it is shown how the three trajectories converge to the same. Logarithmic scale were used in both axes to show the transient phase. In Figure 9 it can be observed how in the three cases, the same error is achieved.

With these two examples, we have illustrated the two main properties of (14). First, the delay size and its rate of variation does not affect directly the ultimate bound for the estimation error. For a system where the output asymptotically remains in a compact region, the error bounds will depend on the size of that region. Second,

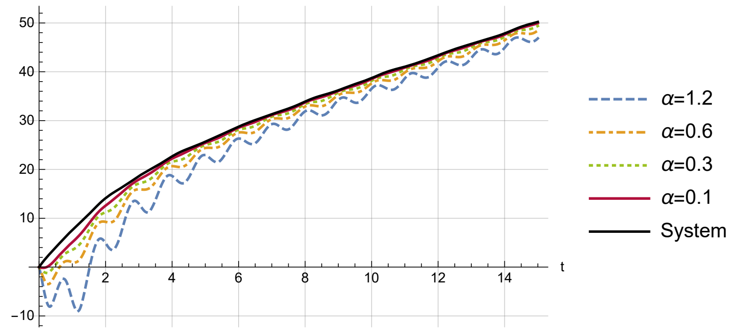

Figure 3. Tracking of the nominal output $\bar{y}(t)$ for the different values of α.

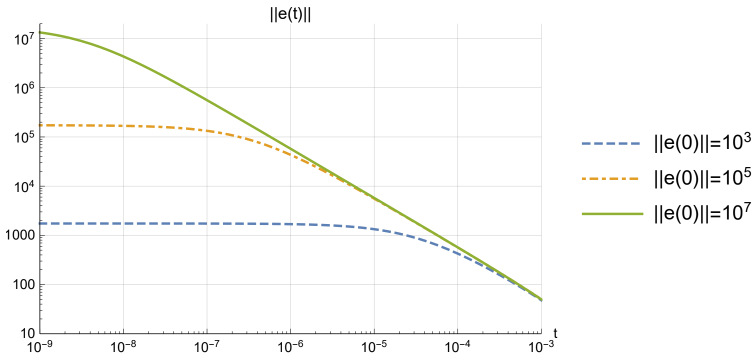

Figure 4. Time evolution of the error norm for different initial errors. In the plot, the fast attraction to the ultimate bound is shown in a logarithmic scale in both axes.

when the estimation error is large, its rate of convergence is accelerated due to the nonlinearity. These properties make the algorithm suitable when the delay and the initial error are uncertain and large.

In the case of a linear time-invariant system, a comparaison of the proposed observer with a delayed Luenberger observer (3) and the undelayed version (5) can be found in (Rueda-Escobedo et al., 2018).

7. Conclusion

In the note, an observer for marginally stable time-varying systems with delayed measurements is presented. The observer provides an estimate that converges to the internal state of the system up to a bounded error. The convergence time can be bounded by a constant independent from the initial error and the initial time, meaning that the ultimate bound of the error is reached in uniform fixed-time. The proposed observer proves to be useful when the knowledge about the delay is scarce, or when it is large with respect to the time-scale of the system. Also, if the convergence time of the observer is crucial for the application, the proposed approach is helpful due to the fixed-time convergence since it guarantees the time needed to trust the estimate. This time can be adjusted by means of the observer parameters.

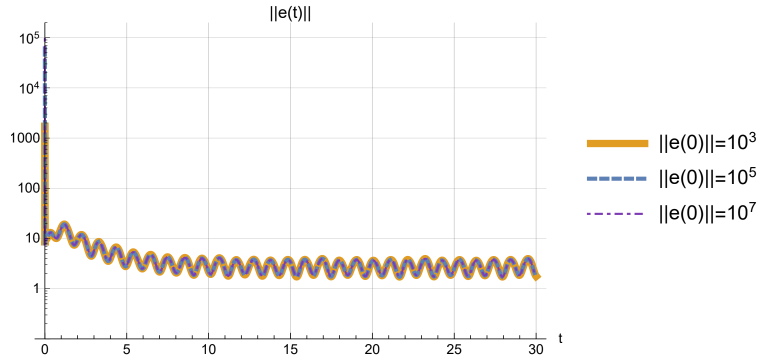

Figure 5. Three trajectories reach the same bounded region despite the differences in the initial condition.

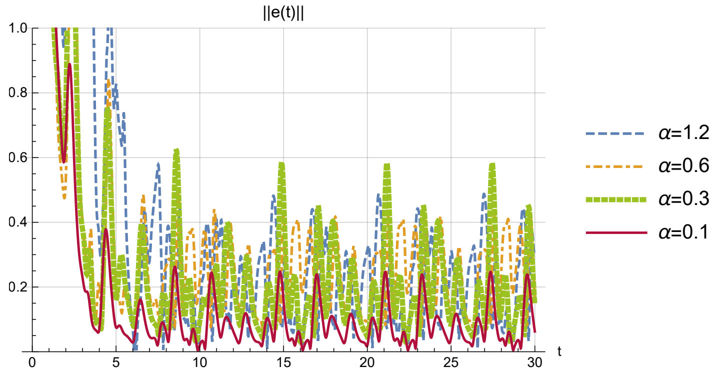

Figure 6. Evolution of the estimation error norm for the different values of α when the system is UAS.

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Appendix A. Some inequalities

Lemma A.1. *Let x, $\delta \in \mathbb{R}$ and $p > 0$. Then, for any $\kappa_1 \in (0, 1)$ there exists $\kappa_2 > 0$ such that*

$$x\lceil x + \delta\rfloor^p \geq \kappa_1 |x|^{p+1} - \kappa_2 |\delta|^{p+1}.$$

In particular, one can select $\kappa_2 = \max\left\{1 + \kappa_1, \, \kappa_1/\left(1 - \kappa_1^{(1/p)}\right)^p\right\}$.

Proof. For $x = 0$ or $\delta = 0$, the inequality is satisfied trivially with any $\kappa_2 \geq 0$, then

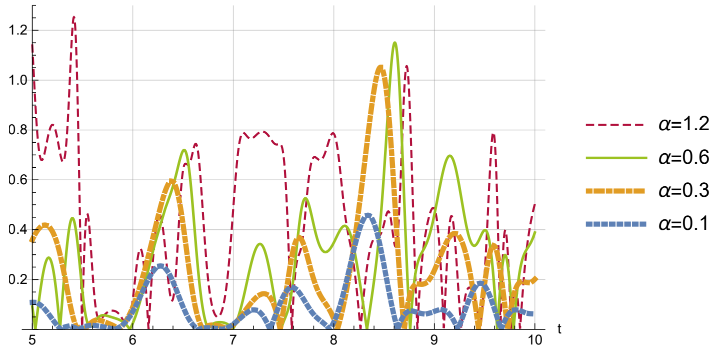

Figure 7. Plot of the difference $|y(t) - \bar{y}(t)|$ when the system is UAS. Increasing the value of α does not affect significantly this difference.

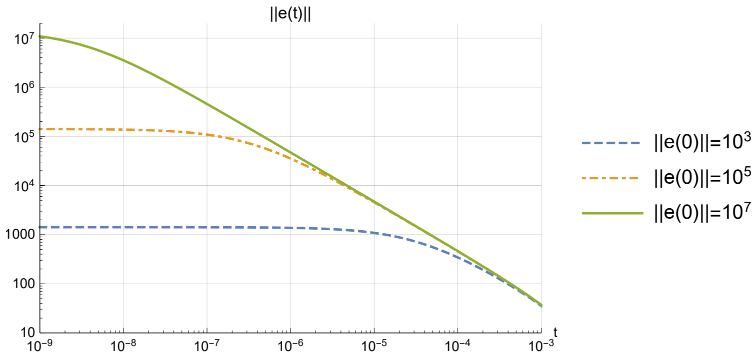

Figure 8. Time evolution of the error norm for different initial errors when the systems is UAS. In the plot, the fast attraction to the ultimate bound is shown in a logarithmic scale in both axes.

consider the case with $x \neq 0$ and $\delta \neq 0$. Now, by homogeneity we have:

$$x\lceil x+\delta\rfloor^p = \frac{1}{\epsilon^{p+1}}\Big((\epsilon\, x)\lceil(\epsilon\, x)+(\epsilon\,\delta)\rfloor^p\Big)$$
$$\kappa_1\,|x|^{p+1} - \kappa_2\,|\delta|^{p+1} = \frac{1}{\epsilon^{p+1}}\Big(\kappa_1\,|\epsilon\, x|^{p+1} - \kappa_2\,|\epsilon\,\delta|^{p+1}\Big),$$

for any $\epsilon > 0$. Set $\epsilon = 1/|\delta|$ and define $z = x/|\delta|$. The inequality then is equivalent to $z\lceil z + \lceil\delta\rfloor^0\rfloor^p \geq \kappa_1\,|z|^{p+1} - \kappa_2$. We will only consider the case $\lceil\delta\rfloor^0 = 1$ since the other one is analogous. For $\lceil\delta\rfloor^0 = 1$ we have $z\lceil z+1\rfloor^p \geq \kappa_1\,|z|^{p+1} - \kappa_2$, or

$$z\lceil z+1\rfloor^p - \kappa_1\,|z|^{p+1} \geq -\,\kappa_2. \tag{A1}$$

This reduces the problem to prove that the left hand side of the inequality has a lower bound. For $z > 0$, we have that $\lceil z+1\rfloor^0 = 1$ and $|z+1| > |z|$, then $z\lceil z+1\rfloor^p > |z|^{p+1}$. Since $\kappa_1 < 1$, $z\lceil z+1\rfloor^p - \kappa_1\,|z|^{p+1} > 0$ and (A1) holds for any $\kappa_2 \geq 0$ on this interval. Now, for $z \in (-1, 0)$, we must show $-|z|\,|z+1|^p - \kappa_1\,|z|^{p+1} \geq -\kappa_2$. In this interval, we have that $|z| < 1$ and $|z+1| < 1$, then $-|z|\,|z+1|^p - \kappa_1\,|z|^{p+1} \geq -1 - \kappa_1$, which implies that (A1) holds with $\kappa_2 \geq 1+\kappa_1$. Last, we consider the interval $z \in (-\infty, -1]$, where now we must check $|z|\,|z+1|^p - \kappa_1|z|^{p+1} \geq -\kappa_2$. Notice that $|z| > |z+1|$ in

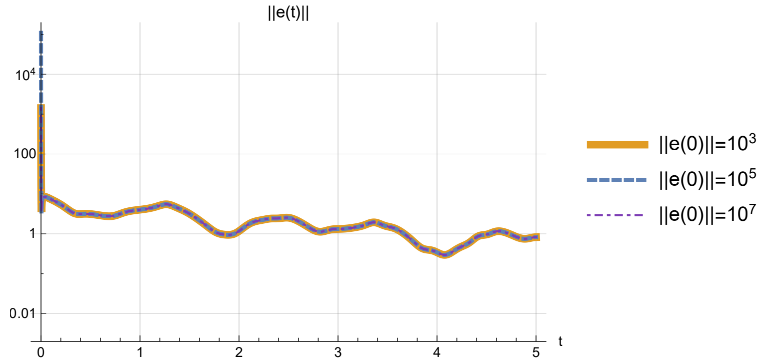

Figure 9. Three trajectories reach the same bounded region despite the differences in the initial condition when the system is UAS.

this case. To find a lower bound, consider the following auxiliary function:

$$|z|\,|z+1|^p - \kappa_1|z|^{p+1} \geq |z+1|^{p+1} - \kappa_1|z|^{p+1} := g(z).$$

Now, we proceed to look for the minimum of $g(z)$. Taking its derivative, this results in $g'(z) = -(p+1)\big(\kappa_1|z|^p - |z+1|^p\big)$, which has a unique zero at $z_0 = -1/(1-\kappa_1^{1/p})$. The second derivative of $g(z)$ evaluated at z_0 is positive, reveling that $g(z)$ has a minimum at this point. Then $g(z_0)$ can be taken as κ_2. This gives us $\kappa_2 \geq \kappa_1/(1-\kappa_1^{1/p})^p$. Finally, looking at the three conditions we get that

$$\kappa_2 \geq \max\left\{\kappa_1+1,\ \frac{\kappa_1}{(1-\kappa_1^{1/p})^p}\right\}.$$

$\square$

Remark 6. In the proof of Lemma A.1, a value for κ_2 is given analytically as a function of κ_1 and p. However, the ratio κ_2/κ_1 can be really large for κ_1 close to one and $p >> 1$. A sharp value for κ_2 can be found numerically by looking at the minimum of $z\lceil z+1\rfloor^p - \kappa_1|z|^{p+1}$ on $z \in (-\infty, 0)$. This is discussed latter in this appendix.

Corollary A.2. *Let x, $\delta \in \mathbb{R}^n$ and $p > 0$. Then, for any $\kappa_1 \in (0,1)$ there is $\kappa_2 > 0$ such that*

$$x^\top \lceil x+\delta\rfloor^p \geq \kappa_1\|x\|_{p+1}^{p+1} - \kappa_2\|\delta\|_{p+1}^{p+1}.$$

In particular, for $p > 1$ we have

$$x^\top \lceil x+\delta\rfloor^p \geq \kappa_1 n^{\frac{1-p}{2}}\|x\|^{p+1} - \kappa_2\|\delta\|^{p+1}.$$

In both cases, κ_2 can be taken as in Lemma A.1.

Proof. First, the product $x^\top\lceil x+\delta\rfloor^p$ is developed and bounded using Lemma A1:

$$x^\top\lceil x+\delta\rfloor^p = \sum_{i=1}^n x_i\lceil x_i+\delta_i\rfloor^p \geq \kappa_1\sum_{i=1}^n |x_i|^{p+1} - \kappa_2\sum_{i=1}^n|\delta_i|^{p+1}.$$

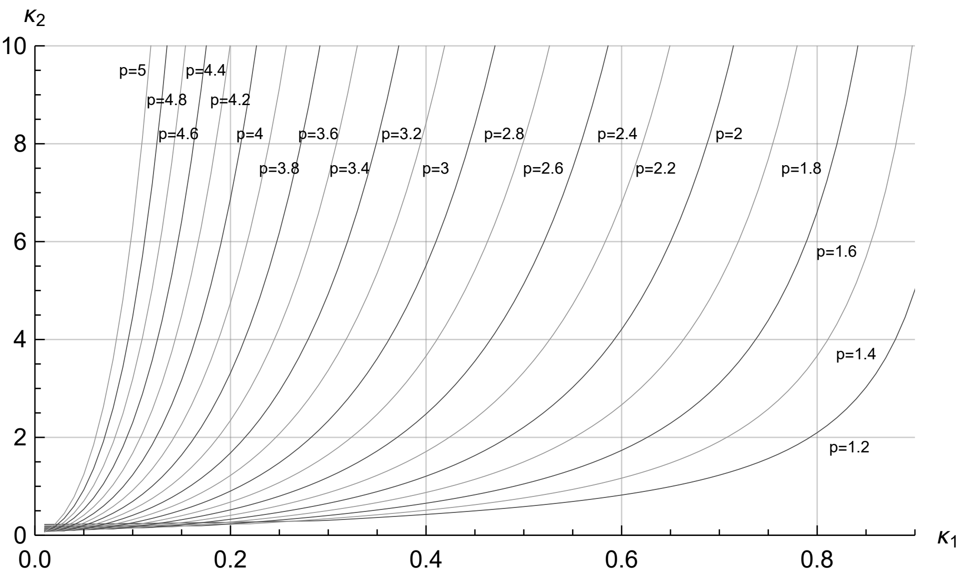

Figure A1. Values of κ_2 for different values of κ_1 and p.

Each of the sum represent the $(p+1)$-norm raised to $p+1$. From here, the first statement of Corollary A.2 follows. Now, $p > 1 \implies p+1 > 2$. Then, using the equivalence between norms in $\mathbb{R}^n$, we have

$$\kappa_1 \|x\|_{p+1}^{p+1} - \kappa_2 \|\delta\|_{p+1}^{p+1} \geq \kappa_1 n^{\frac{1-p}{2}} \|x\|^{p+1} - \kappa_2 \|\delta\|^{p+1}.$$

This concludes the proof. □

A.1. Discussion

To obtain a sharp value for κ_2, the minimum of $f(z) := z\lceil z+1 \rfloor^p - \kappa_1 |z|^{p+1}$ on the interval $(-\infty, 0)$ is required. This can be done by looking at its derivative:

$$\begin{aligned}
f'(z) &= \lceil z+1 \rfloor^p + p\, z \lceil z+1 \rfloor^{p-1} - (p+1)\kappa_1 \lceil z \rfloor^p, \\
f'(z) &= |z+1|^p + (p+1)\kappa_1 |z|^p - p\,|z|\,|z+1|^{p-1},\ z \in (-1, 0), \\
f'(z) &= (p+1)\kappa_1 |z|^p - |z+1|^p - p\,|z|\,|z+1|^{p-1},\ z \in (-\infty, -1].
\end{aligned}$$

Since $|z+1|^p$, $|z|^p$, and $|z|\,|z+1|^{p-1}$ are monotonically increasing functions of z, there are two critical points, one in each sub-interval. The zeros of $f'(z)$ can be found numerically and then used to find the minimum for $f(z)$. In Figure A1, we present the values of κ_2 found using this approach for some values of p.

Beside the value of κ_1 and κ_2, one may be interested in the best selection of these parameters for a given p. To chose some $\kappa_1^\star$ and its associated $\kappa_2^\star$, we propose to take the ones that minimize the ratio $\left(\kappa_2/\kappa_1\right)^{\frac{1}{p+1}}$. From the data generated for Figure A1 we got the values presented in Table A1 and A2.

Table A1. Suggested values of κ_1 and κ_2 for some p.

p	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8
κ_1	0.297	0.219	0.173	0.1407	0.1162	0.097	0.08134	0.0685	0.0579
κ_2	0.300	0.232	0.1946	0.1703	0.1529	0.1396	0.1288	0.1199	0.1123
$(\kappa_2/\kappa_1)^{\frac{1}{p+1}}$	1.005	1.024	1.046	1.071	1.096	1.121	1.145	1.168	1.190

Table A2. Suggested values of κ_1 and κ_2 for some p, continuation.

p	3	3.4	3.8	4	4.4	4.8	5
κ_1	0.049	0.0353	0.0256	0.0218	0.0159	0.0117	0.00997
κ_2	0.1057	0.0946	0.0858	0.0819	0.0757	0.0697	0.0671
$(\kappa_2/\kappa_1)^{\frac{1}{p+1}}$	1.212	1.251	1.286	1.303	1.333	1.361	1.374

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